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The quintic finite element and finite strip with generalized degrees of freedom in structural analysis

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Abstract

This paper presents the method of quintic finite element and quintic finite strip with generalized degrees of freedom (DOF) based on the philosophy that the local displacement fields of an element or strip should be compatible with the global displacement field. At first, the global displacement field of a structure is developed using the quintic B-spline functions. Then, the local displacement field of element/strip is constructed employing the multi-term interpolation functions of degree 5. At last, the displacement field of finite element/strip with generalized DOF is generated when the local and global displacement fields are equally matched in their deformation. This kind of high order finite element/strip can not only effectively widen the scope of application of the conventional quintic element/strip and the spline function method, but also greatly reduce the amount of DOF with the same degrees of accuracy as compared to the conventional finite element method. Several numerical examples demonstrate the accuracy, simplicity and versatility of the present methods for analysis of thin-walled structures.

Keywords: Spline function; Finite element method; Generalized degrees of freedom

1. Introduction

In numerical analysis of engineering structures, finite element method (FEM) (Rao, 1976) is undoubtedly the most powerful tool for its accuracy and versatility. Researchers have kept on working at FEM to widen the scope of its applications, and try to reduce its degrees of freedom (DOF). Finite strip method (FSM) (Cheung, 1976) is one of the famous semi-analytical numerical methods combining conventional analytical method with the FEM. The displacement field of FSM for plate bending problems consists of multi-term polynomial interpolation in the longitudinal direction multiplied by continuous series in the transverse direction. Therefore, a two-dimensional discretization problem of FEM will degenerate into a one-dimensional discretization problem of FSM because of the usage of continuous series in the transverse direction. As a result, the number of DOF in FSM are usually far less than that in FEM.

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The incorporation of spline functions with the conventional FEMs is another remarkable progress achieved, a typical example is the spline finite element method (SFEM) presented by Shi (1979). In the SFEM, the cubic B-spline functions are introduced into the displacement field in both longitudinal and transverse directions, when geometry of the structure under consideration is regular, as is the case for many practical structures. Because the spline functions are piecewise polynomials, which have the property of high continuity, the SFEM compares favorably with the conventional FEM in terms of the number of DOF, computational time, storage requirement and the ease of date preparation and input.

In consideration of the merits of the spline function methods and the FSM, the proper combination of these two effective tools will surely provide promising numerical approaches for structural analysis. Cheung et al. (1982) developed the spline finite strip method (SFSM), in which the displacement field is constructed by means of interpolation polynomials in the longitudinal direction multiplied by the cubic B-spline functions in the transverse directions. Qin (1981) presented the spline point method (SPM), which defines its displacement field using the continuous series in transverse direction multiplied by the spline functions in the longitudinal direction.

But the numerical methods associated with the spline functions suffer from the fact that their applications to structural analysis is cumbersome when some coefficients such as Young's modulus, Poisson's ratio or structural thickness vary along the direction in which the spline functions are interpolated and defined. That is to say, the applications of the conventional spline function methods have been limited in the problems with regular geometry and physical coefficients.

The higher order finite element/strip, e.g. quintic one, is popularly used, and can greatly reduce the computation time when the gradient of the field variable is expected to vary rapidly. And in this case the lower order elements/strips which approximate the gradient with a set of constant values will not yield reasonable results. In other cases, the higher order element/strip method will not result in a reduction of computational time though it can greatly reduce the number of elements/strips with the same degrees of accuracy, because a higher order element/strip has much more DOF than a lower one. Another fatal drawback of the quintic finite element/strip, which contains the second partial derivatives of the field variable, is that it is not convenient for this kind of element to use in the analysis of plates or beams with variable thickness on account of the discontinuity of the curvatures.

In order to overcome the aforementioned drawbacks of the conventional spline function methods and the higher order element/strip, a finite element/strip with generalized DOF is presented in this paper. The displacement field of this element/strip is constructed by means of the combination of the quintic spline functions and the second-order Hermitian functions through the following procedures. At first, the spline functions are employed to construct the global displacement field of interest, as the SFEM or SFPM does. Secondly, the local displacement field of the element/strip is constructed using the multi-term interpolation polynomials, as the FEM or FSM does. At last a new kind of local displacement field of the element with generalized DOF is derived when the local and global displacement fields are equally matched in their deformation.

2. Formulation of the element with generalized coefficients

2.1. Quintic B-spline function

An arbitrary division of a given interval $[a, b]$ is defined as follows:

$$\Delta: \quad a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

One can construct the quintic B-spline functions corresponding to the division Δ:

$$B_{5,i}(x) = (x_{i+3} - x_{i-3}) \sum_{k=i-3}^{i+3} \frac{(x_k - x)_+^5}{W'_{5,i}(x_k)}, \quad \text{for } i = -2, -1, 0, 1, \ldots, n+2 \tag{1}$$

where

$$W_{5,i}(x) = \prod_{j=-3}^{3} (x - x_{i+j}) \tag{2}$$

$$(x_k - x)_+ = \max\{0, (x_k - x)\} \tag{3}$$

2.2. Interpolation function of the global field of interest

At first, an elastic beam shown in Fig. 1 is considered as an example to demonstrate the formulation of the element with generalized DOF for the sake of simplicity.

A subdivision of the beam with length l, and Young's modulus E, carrying external loading $q(x)$, is defined as Δ on condition that $a = 0, \ b = l$. Therefore the deflection of the beam can be approximated by the interpolation quintic B-spline functions as follows.

$$w(x) = \sum_{i=-2}^{n+2} \Phi_i(x) c_i = [\Phi]\{c\} = [B][Q]\{c\} \tag{4}$$

where

$$\begin{aligned} &[\Phi] = [\Phi_{-2}, \Phi_{-1}, \Phi_0, \Phi_1, \ldots, \Phi_{n+2}] \\ &[B] = [B_{-2}, B_{-1}, B_0, \ldots, B_{n+2}] \\ &\{c\} = [w_0, w'_0, w''_0, c_1, \ldots, c_{n-1}, w_n, w'_n, w''_n]^{\mathrm{T}} \\ &[\Phi] = [B][Q] \end{aligned} \tag{5}$$

in which, B_i for $i = -1, 0, 1, \ldots, n+1$ denote the quintic B-spline functions, $B_{5,i}(x)$, as shown in Eq. (1). c_i for $i = 1, 2, \ldots, n-1$ is an unknown coefficient at node x_i, which has no specific geometrical or physical meanings. w_0 and w_n are displacements of the beam at $x = 0$ and $x = x_n = l$, respectively. w'_0 and w'_n denote the first derivatives of the displacement of the beam with respect to x at $x = 0$ and $x = x_n = l$, respectively. w''_0 and w''_n denote the second derivatives of the displacement of the beam with respect to x at

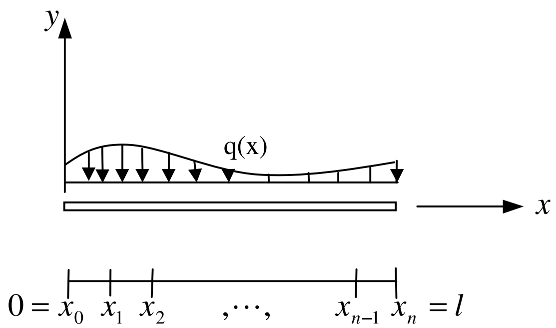

Fig. 1. Elastic beam and division.

$x = 0$ and $x = x_n = l$, respectively. And function Φ_i for $i = -2, -1, 0, n-1, n+1, n+2$, is of a linear combination of the quintic B-spline functions in terms of the relationship of Φ_i and B_i defined in Eq. (5). For $i = 1, 2, \ldots, n-1$, Φ_i is the same as B_i when $n \geqslant 3$.

When Δ is defined as a uniform division Δ1, i.e.

$$\Delta 1: \quad a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

$$x_i = x_0 + \mathrm{i}h \qquad h = \frac{b-a}{n} \tag{6}$$

Here, the conditions $a = 0,\ b = l$ should be also satisfied for the beam as aforementioned.

The quintic B-spline functions have the following simplified explicit form corresponding to the uniform division Δ1:

$$B_{5,i}(x) = \frac{1}{120h^5}\begin{cases} 0, & x \notin [x_{i-3}, x_{i+3}] \\ (x - x_{i-3})^5, & x \in [x_{i-3}, x_{i-2}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5, & x \in [x_{i-2}, x_{i-1}] \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5, & x \in [x_{i-1}, x_i] \\ (x_{i+3} - x)^5 - 6(x_{i+2} - x)^5 + 15(x_{i+1} - x)^5, & x \in [x_i, x_{i+1}] \\ (x_{i+3} - x)^5 - 6(x_{i+2} - x)^5, & x \in [x_{i+1}, x_{i+2}] \\ (x_{i+3} - x)^5, & x \in [x_{i+2}, x_{i+3}] \end{cases} \tag{7}$$

$B_i(x)$ shown in Eq. (7) will lead to the corresponding $[Q]$ as follows:

$$[Q] = \mathrm{diag}([Q_0], [I], [Q_n]) \tag{8}$$

where $[I]$ denotes a unit square matrix of order $(n-5) \times (n-5)$ when $n \geqslant 5$. Both $[Q_0]$ and $[Q_n]$ are square matrices of order 5×5, shown as follows:

$$[Q_0] = \frac{1}{36}\begin{bmatrix} 600 & 576h & 330h^2 & -480 & -84 \\ -60 & -144h & -33h^2 & 84 & 12 \\ 80 & 48h & 8h^2 & -40 & -4 \\ 0 & 0 & 0 & 36 & 0 \\ 0 & 0 & 0 & 0 & 36 \end{bmatrix} \tag{9a}$$

$$[Q_n] = \frac{1}{36}\begin{bmatrix} 36 & 0 & 0 & 0 & 0 \\ 0 & 36 & 0 & 0 & 0 \\ -4 & -40 & 80 & -48h & 8h^2 \\ 12 & 84 & -60 & 144h & -33h^2 \\ -84 & -480 & 600 & -576h & 330h^2 \end{bmatrix} \tag{9b}$$

If $n = 1$, $[Q]$ should be:

$$[Q] = \frac{1}{20}\begin{bmatrix} 2460 & 1328h & 329h^2 & -2440 & 1072h & -146h^2 \\ -440 & -272h & -46h^2 & 460 & -208h & 29h^2 \\ 260 & 128h & 19h^2 & -240 & 112h & -16h^2 \\ -240 & -112h & -16h^2 & 260 & -128h & 19h^2 \\ 460 & 208h & 29h^2 & -440 & 272h & -46h^2 \\ -2440 & -1072h & -146h^2 & 2460 & -1328h & 329h^2 \end{bmatrix} \tag{10}$$

For $n = 2$, $[Q]$ reads:

$$[Q] = \begin{bmatrix} 22.5 & 19.5h & (39/4)h^2 & -11 & 0 & 0 & 0 \\ -(119/44) & -(1017/220)h & -(449/440)h^2 & 2 & -(9/44) & (27h/220) & -(9h^2/440) \\ (30/11) & (18h/11) & (3h^2/11) & -1 & (5/22) & -(3h/22) & (h^2/44) \\ -(21/44) & -(63h/220) & -(21h^2/440) & 1 & -(21/44) & (63h/220) & -(21h^2/440) \\ (5/22) & (3h/22) & (h^2/44) & -1 & (30/11) & -(18h/11) & (3h^2/11) \\ -(9/44) & -(27h/220) & -(9h^2/440) & 2 & -(119/44) & (1017/220)h & -(449/440)h^2 \\ 0 & 0 & 0 & -11 & (45/2) & -(39/2)h & (39/4)h^2 \end{bmatrix} \tag{11}$$

For $n = 3$ or 4, $[Q]$ is also composed of $[Q_0]$ and $[Q_n]$, and the elements 36 on diagonals of these two matrices will be overlapped. For example, $n = 4$ will lead to

$$[Q] = \frac{1}{36}\begin{bmatrix} 600 & 576h & 330h^2 & -480 & -84 & 0 & 0 & 0 & 0 \\ -60 & -144h & -33h^2 & 84 & 12 & 0 & 0 & 0 & 0 \\ 80 & 48h & 8h^2 & -40 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 36 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 36 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & -40 & 80 & -48h & 8h^2 \\ 0 & 0 & 0 & 0 & 12 & 84 & -60 & 114h & -33h^2 \\ 0 & 0 & 0 & 0 & -84 & -480 & 600 & -576h & 330h^2 \end{bmatrix}$$

$[Q]$ is named cardinal transformation matrix (Yang, 1998a), which makes Φ_i, for $i = -1, 0, n, n+1$ in Eq. (5) keep the cardinal property.

$$\Phi_j(x_0) = \begin{cases} 1 & \text{if } j = -2 \\ 0 & \text{if } j \neq -2 \end{cases}, \quad \Phi'_j(x_0) = \begin{cases} 1 & \text{if } j = -1 \\ 0 & \text{if } j \neq -1 \end{cases}, \quad \Phi''_j(x_0) = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{if } j \neq 0 \end{cases}$$
$$\Phi_j(x_n) = \begin{cases} 1, & \text{if } j = n \\ 0, & \text{if } j \neq n \end{cases}, \quad \Phi'_j(x_n) = \begin{cases} 1, & \text{if } j = n+1 \\ 0, & \text{if } j \neq n+1 \end{cases}, \quad \Phi''_j(x_n) = \begin{cases} 1, & \text{if } j = n+2 \\ 0, & \text{if } j \neq n+2 \end{cases} \tag{12}$$

Thus, the prescribed boundary conditions are convenient to incorporate, following procedures of the standard FEM. And the localized nature of the B-spline functions is retained at all the knots except x_i $(i = 0, 1, 2, n-2, n-1, n)$ in interval $[a, b]$, so that the stiffness matrix remains sparse.

It should be noted that there is only one DOF per inner node in the global displacement field of interest. This is usually the case even if the displacement field is developed by means of the lower order spline functions. So the number of DOF of the quintic global displacement field only slightly increases, as compared with the lower order global displacement field.

2.3. Local displacement field of typical element

The segment $[x_{e-1}, x_e]$ of the beam is called element e, for $e = 1, 2, \ldots, n$. The displacement model of this element can be constructed in terms of the methodology of the conventional quintic finite element:

$$w^e = \sum_{i=1}^{6} H_i(x)\delta_i^e = [H]\{\delta\}^e \tag{13}$$

where

$$[H] = [H_1, H_2, H_3, H_4, H_5, H_6]$$

$$\{\delta\}^e = [\delta_1^e, \delta_2^e, \delta_3^e, \delta_4^e, \delta_5^e, \delta_6^e]^{\mathrm{T}}$$

in which H_i, $(i = 1, 2, \ldots, 6)$, denoting $H_i(x)$, are multi-term polynomial functions of fifth degree. Here the second-order hermitian functions are adopted:

$$\begin{aligned} &H_1(x) = 1 - 10\xi^3 + 15\xi^4 - 6\xi^5, \qquad H_4(x) = 10\xi^3 - 15\xi^4 + 6\xi^5 \\ &H_2(x) = (\xi - 6\xi^3 + 8\xi^4 - 3\xi^5)h, \qquad H_5(x) = (-4\xi^3 + 7\xi^4 - 3\xi^5)h \\ &H_3(x) = \frac{1}{2}(\xi^2 - 3\xi^3 + 3\xi^4 - \xi^5)h^2, \qquad H_6(x) = \frac{1}{2}(\xi^3 - 2\xi^4 + \xi^5)h^2 \end{aligned} \tag{14}$$

where

$$\xi = \frac{x - x_{e-1}}{h}$$

In Eq. (13), δ_i^e, $(i = 1, 2, \ldots, 6)$, denotes DOF of element e, which has specific meanings as follows:

$$\begin{aligned} &\delta_1^e = w^e(x_{e-1}), \qquad \delta_2^e = \frac{\mathrm{d}w^e(x_{e-1})}{\mathrm{d}x}, \qquad \delta_3^e = \frac{\mathrm{d}^2 w^e(x_{e-1})}{\mathrm{d}x^2} \\ &\delta_4^e = w^e(x_e), \qquad \delta_5^e = \frac{\mathrm{d}w^e(x_e)}{\mathrm{d}x}, \qquad \delta_6^e = \frac{\mathrm{d}^2 w^e(x_e)}{\mathrm{d}x^2} \end{aligned} \tag{15}$$

This is a conventional quintic finite element.

2.4. Displacement field of the element with generalized degrees of freedom

As aforementioned, the local displacement field w^e of the element must be consistent with the global displacement field of interest, $w(x)$. That is to say, values of global displacement field and its derivatives with respect to x should be equal to those of the local displacement field of the element, respectively, at all nodes of the element. Hence, $w^e(x)$ and its first and second derivatives with respect to x in Eq. (15) can be replaced by $w(x)$ and its first and second derivatives with respect to x, respectively. Accordingly, Eq. (15) can be expressed in a matrix form as:

$$\{\delta\}^e = \begin{Bmatrix} \delta_1^e \\ \delta_2^e \\ \delta_3^e \\ \delta_4^e \\ \delta_5^e \\ \delta_6^e \end{Bmatrix} = \begin{Bmatrix} w(x_{e-1}) \\ w'(x_{e-1}) \\ w''(x_{e-1}) \\ w(x_e) \\ w'(x_e) \\ w''(x_e) \end{Bmatrix} \tag{16}$$

Substituting Eq. (4) into Eq. (16), one has,

$$\{\delta\}^e = \begin{Bmatrix} w(x_{e-1}) \\ w'(x_{e-1}) \\ w''(x_{e-1}) \\ w(x_e) \\ w'(x_e) \\ w''(x_e) \end{Bmatrix} = \begin{Bmatrix} [B(x_{e-1})] \\ [B'(x_{e-1})] \\ [B''(x_{e-1})] \\ [B(x_e)] \\ [B'(x_e)] \\ [B''(x_e)] \end{Bmatrix} [Q]\{c\} \tag{17}$$

For the sake of clarity, transformation matrix $[Q]$ is neglected in the following process of derivation in consideration that both $[Q]$ and $\{c\}$ are constant matrices, and accordingly, the product of $[Q]$ and $\{c\}$ can be regarded as a new vector.

The quintic B-spline functions and their derivatives, $B_{5,i}(x)$, $B'_{5,i}(x)$ and $B''_{5,i}(x)$ will vanish when $x \notin [x_i - 3h, x_i + 3h]$. Therefore, Eq. (17) can be expressed as:

$$\{\delta\}^e = [T]\{c\}^e \tag{18}$$

where

$$\{c\}^e = [c_1^e, c_2^e, c_3^e, c_4^e, c_5^e, c_6^e]^{\mathrm{T}} = [c_{e-3}, c_{e-2}, c_{e-1}, c_e, c_{e+1}, c_{e+2}]^{\mathrm{T}} \tag{19}$$

$$[T] = \begin{bmatrix} B_{e-3}(x_{e-1}) & B_{e-2}(x_{e-1}) & B_{e-1}(x_{e-1}) & B_e(x_{e-1}) & B_{e+1}(x_{e-1}) & 0 \\ B'_{e-3}(x_{e-1}) & B'_{e-2}(x_{e-1}) & B'_{e-1}(x_{e-1}) & B'_e(x_{e-1}) & B'_{e+1}(x_{e-1}) & 0 \\ B''_{e-3}(x_{e-1}) & B''_{e-2}(x_{e-1}) & B''_{e-1}(x_{e-1}) & B''_e(x_{e-1}) & B''_{e+1}(x_{e-1}) & 0 \\ 0 & B_{e-2}(x_e) & B_{e-1}(x_e) & B_e(x_e) & B_{e+1}(x_e) & B_{e+2}(x_e) \\ 0 & B'_{e-2}(x_e) & B'_{e-1}(x_e) & B'_e(x_e) & B'_{e+1}(x_e) & B'_{e+2}(x_e) \\ 0 & B''_{e-2}(x_e) & B''_{e-1}(x_e) & B''_e(x_e) & B''_{e+1}(x_e) & B''_{e+2}(x_e) \end{bmatrix} \tag{20}$$

Eq. (19) shows the location of vector $\{c\}^e$ in vector $\{c\}$ defined in Eq. (4).

When the division of the beam defined as a uniform one, $\Delta 1$, the explicit form of $[T]$ is given below:

$$[T] = \frac{1}{120}\begin{bmatrix} 1 & 26 & 66 & 26 & 1 & 0 \\ (5/h) & (50/h) & 0 & -(50/h) & -(5/h) & 0 \\ (20/h^2) & (40/h^2) & -(120/h^2) & (40/h^2) & (20/h^2) & 0 \\ 0 & 1 & 26 & 66 & 26 & 1 \\ 0 & (5/h) & (50/h) & 0 & -(50/h) & -(5/h) \\ 0 & (20/h^2) & (40/h^2) & -(120/h^2) & (40/h^2) & (20/h^2) \end{bmatrix} \tag{21}$$

Substituting Eq. (18) into Eq. (13), one has:

$$w^e = [H]\{\delta\}^e = [H][T]\{c\}^e = [N]\{c\}^e \tag{22}$$

where

$$[N] = [H][T] = [N_1, N_2, N_3, N_4, N_5, N_6] \tag{23}$$

When the division of the beam is the uniform one $\Delta 1$, N_i $(i = 1, 2, \ldots, 6)$ can be derived by means of substituting Eq. (14) and Eq. (21) into Eq. (23) as follows:

$$\begin{aligned} N_1(x) &= \frac{1}{120}(1 - \xi^5) \\ N_2(x) &= \frac{1}{120}(26 - 50\xi + 20\xi^2 + 20\xi^3 - 20\xi^4 + 5\xi^5) \\ N_3(x) &= \frac{1}{60}(33 - 30\xi^2 + 15\xi^4 - 5\xi^5) \\ N_4(x) &= \frac{1}{60}(13 + 25\xi + 10\xi^2 - 10\xi^3 - 10\xi^4 + 5\xi^5) \\ N_5(x) &= \frac{1}{120}(1 + 5\xi + 10\xi^2 + 10\xi^3 + 5\xi^4 - 5\xi^5) \\ N_6(x) &= \frac{1}{120}5\xi^5 \end{aligned} \tag{24}$$

Eq. (22) is the local displacement field of the element with generalized DOF, and all the nodal parameters c_i^e, for $i = 1, 2, \dots, 6$, do not have any physical or geometrical meanings.

2.5. Stiffness equations of the element with generalized degrees of freedom

When the functional of potential energy is developed

$$\prod^e = \frac{1}{2}\{c\}^{e\mathrm{T}}[G]^e\{c\}^e - \{c\}^{e\mathrm{T}}\{P\}^e \tag{25}$$

stiffness equations of the element with generalized DOF can be readily constructed by means of the variational principle:

$$[G]^e\{c\}^e = \{P\}^e \tag{26}$$

where

$$[G]^e = \int_0^1 D^e[N'']^{\mathrm{T}}[N'']h\,\mathrm{d}\xi, \qquad \{P\}^e = \int_0^1 [N]^{\mathrm{T}} q^e h\,\mathrm{d}\xi \tag{27}$$

in which both D^e and q^e are often regarded as constants when they do not change abruptly in the region of element e. So one has:

$$[G]^e = D^e[B^{22}], \qquad \{P\}^e = q^e\{f\} \tag{28}$$

where

$$[B^{22}] = \int_0^1 [N'']^{\mathrm{T}}[N'']h\,\mathrm{d}\xi, \qquad \{f\} = \int_0^1 [N]^{\mathrm{T}} h\,\mathrm{d}\xi \tag{29}$$

Substituting Eq. (24) into Eq. (29), $[B^{11}]$and$\{f\}$ can be derived:

$$[B^{22}] = \frac{1}{5040h^3}\begin{bmatrix} 20 & 89 & -178 & 10 & 58 & 1 \\ & 752 & -958 & -638 & 697 & 58 \\ & & 1832 & -68 & -638 & 10 \\ & & & 1832 & -958 & -178 \\ & & \text{symm} & & 752 & 89 \\ & & & & & 20 \end{bmatrix}; \qquad \{f\} = \frac{h}{720}\begin{Bmatrix} 1 \\ 57 \\ 302 \\ 302 \\ 57 \\ 1 \end{Bmatrix} \tag{30}$$

The functional $\prod$ associated with the beam is the sum of the potential energy $\prod^e$ of all elements:

$$\prod = \sum_e \prod^e = \frac{1}{2}\{c\}^{\mathrm{T}}[G]\{c\} - \{c\}^{\mathrm{T}}\{P\} \tag{31}$$

where $[G]$ and $\{P\}$ are global characteristic matrix and global characteristic vector, respectively, which are generated by means of assembling all the element characteristic matrices $[G]^e$ and the element characteristic vectors $\{P\}^e$, respectively. Locations of the element stiffness matrices in the global stiffness matrix can be identified in terms of the relationship between the global vector $\{c\}$ and the local vector $\{c\}^e$ shown in Eq. (19).

Now, we should incorporate transformation matrix $[Q]$, which was neglected for the sake of simplicity as aforementioned, into Eq. (31). That is to say, vector $\{c\}$ should be replaced by $([Q]\{c\})$ in Eq. (31):

$$\prod = \sum_e \prod^e = \frac{1}{2}\{c\}^{\mathrm{T}}[Q]^{\mathrm{T}}[G][Q]\{c\} - \{c\}^{\mathrm{T}}[Q]^{\mathrm{T}}\{P\} = \frac{1}{2}\{c\}^{\mathrm{T}}[K]\{c\} - \{c\}^{\mathrm{T}}\{F\} \tag{32}$$

in which:

$$[K] = [Q]^{\mathrm{T}}[G][Q], \qquad \{F\} = [Q]^{\mathrm{T}}\{P\} \tag{33}$$

Based on the variational principle, the characteristic equations of the beam can be derived from Eq. (32):

$$[K]\{c\} = \{F\} \tag{34}$$

Through the process of assembly of the element characteristic matrices, it can be concluded that the number of DOF of the present method depends on the number of DOF of the global field rather than the local one.

3. Finite strip with generalized degrees of freedom

The conventional FSM is a powerful tool in analysis of structures with regular geometry. This method constructs the displacement field of, for example, an elastic thin plate employing continuous series in the transverse direction and standard polynomial interpolation functions in the longitudinal direction.

In this paper, a new set of interpolation functions N_i $(i = 1, 2, \ldots, 6)$ derived in Section 2.4 is used as the substitution of the standard second-order hermitian functions to construct the displacement field of the finite strip with generalized DOF for analysis of elastic thin plates with constant or variable thickness. The procedures are illustrated as follows.

3.1. Interpolation function of the global field of plate's displacement

An elastic thin plate is shown in Fig. 2, with dimensions $l_x \times l_y$, Young's modulus E, Poisson's ratio v, and carrying external loading q. When there exists no abrupt change of the geometry or the elastic coefficients, the subdivision of the plate in x direction should be defined as $\Delta 1$ on condition that $a = 0$, $b = l_x$ in Eq. (6). Otherwise the subdivision should be defined as a non-uniform one, $\varDelta$, in term of the geometry and material property of the plate. So the deflection function of the plate can be approximated by the quintic B-spline functions using SFPM or the spline Ritz method (Yang, 1998b):

$$w(x, y) = \sum_{m=1}^{r} w_m(x, y) = \sum_{m=1}^{r} [B][Q]\{c\}_m Y_m(y) = \sum_{m=1}^{r} [\overline{N}]_m \{c\}_m = [\overline{N}]\{c\} \tag{35}$$

where $[B]$ and $[Q]$ are given in Eq. (4), and

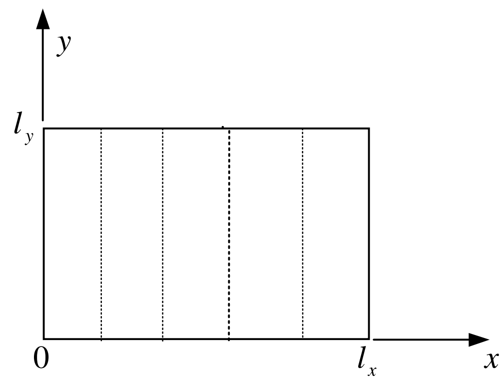

Fig. 2. An elastic plate divided into strips.

$$\begin{aligned}[\overline{N}] &= \left[[\overline{N}]_1, [\overline{N}]_2, \dots, [\overline{N}]_r\right] \\ [\overline{N}]_m &= [[B]Y_1, [B]Y_2, \dots, [B]Y_r]\end{aligned} \tag{36}$$

$$\begin{aligned}\{c\} &= \left[\{c\}_1^{\mathrm{T}}, \{c\}_2^{\mathrm{T}}, \dots, \{c\}_r^{\mathrm{T}}\right]^{\mathrm{T}} \\ \{c\}_m &= [w_{0,m}, w'_{0,m}, w''_{0,m}, c_{1,m}, c_{2,m}, \dots, c_{n-1,m}, w_{n,m}, w'_{n,m}, w''_{n,m}]^{\mathrm{T}}\end{aligned} \tag{37}$$

in which, $Y_m(y)$, denoted by Y_m in the row matrix $[N]_m$, is a trigonometrical series satisfying the boundary conditions of the plate in y-direction, $c_{i,m}$ for $m = 1, 2, \dots, r$ and $i = 1, 2, \dots, n-1$, are unknown coefficients at nodal line x_i, corresponding to the series term r, which have no geometrical or physical meanings. The multiplication of $w_{0,m}$ by the series Y_m, denotes the displacement of the plate at the end nodal line $x = 0$ corresponding to the series term m. The multiplication of $w_{n,m}$ by the series Y_m, denotes the displacement at the end nodal line $x = l$. Similarly, the derivatives $w'_{0,m}$, $w'_{n,m}$, $w''_{0,m}$ and $w''_{n,m}$ have their specific meanings, respectively. So the prescribed boundary conditions can be incorporated conveniently.

3.2. Local displacement field of typical strip

A finite strip e consists of a region of the plate between nodal line x_{e-1} and x_e, $e = 1, 2, \dots, n$. The local displacement field of this strip can be constructed according to the philosophy of conventional FSM:

$$w^e = \sum_{m=1}^{r} w_m^e = \sum_{m=1}^{r} [H] Y_m(y) \{\delta\}_m^e = [\overline{H}]\{\delta\}^e \tag{38}$$

where,

$$\begin{aligned}[\overline{H}] &= \left[[\overline{H}]_1, [\overline{H}]_2, \dots, [\overline{H}]_r\right] \\ [\overline{H}]_m &= [H] Y_m(y)\end{aligned} \tag{39}$$

$$\begin{aligned}\{\delta\}^e &= \left[\{\delta\}_1^{e\mathrm{T}}, \{\delta\}_2^{e\mathrm{T}}, \dots, \{\delta\}_r^{e\mathrm{T}}\right]^{\mathrm{T}} \\ \{\delta\}_m^{e\mathrm{T}} &= [\delta_{1,m}^e, \delta_{2,m}^e, \delta_{3,m}^e, \delta_{4,m}^e, \delta_{5,m}^e, \delta_{6,m}^e]\end{aligned} \tag{40}$$

in which $[H]$ is a row vector of the second-order hermitian function given in Eq. (14). $\delta_{i,m}^e$, for $m = 1, 2, \dots, r$, are DOF of the strip, which have specified meanings.

3.3. Displacement field of the strip with generalized coefficients

w^e and its partial derivatives with respect to x must be equal to $w(x)$ and its partial derivatives with respect to x, respectively, on all the nodal lines of the typical strip e, corresponding to every term of the series, according to the principle that the local displacement field w^e must be compatible with the global displacement field $w(x)$. i.e.

$$\begin{aligned}&w_m^e(x_{e-1}, y) = w_m(x_{e-1}, y), \qquad w_m^e(x_e, y) = w_m(x_e, y) \\ &\frac{\mathrm{d}w_m^e(x_{e-1}, y)}{\mathrm{d}x} = \frac{\mathrm{d}w_m(x_{e-1}, y)}{\mathrm{d}x}, \qquad \frac{\mathrm{d}w_m^e(x_e, y)}{\mathrm{d}x} = \frac{\mathrm{d}w_m(x_e, y)}{\mathrm{d}x} \\ &\frac{\mathrm{d}^2 w_m^e(x_{e-1}, y)}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 w_m(x_{e-1}, y)}{\mathrm{d}x^2}, \qquad \frac{\mathrm{d}^2 w_m^e(x_e, y)}{\mathrm{d}x^2} = \frac{\mathrm{d}^2 w_m(x_e, y)}{\mathrm{d}x^2}\end{aligned} \tag{41}$$

Substitution of Eqs. (35) and (38) into Eq. (41) will lead to the following equation in a form of matrix,

$$\{\delta\}^e = \begin{Bmatrix} [B(x_{e-1})] \\ [B'(x_{e-1})] \\ [B''(x_{e-1})] \\ [B(x_e)] \\ [B'(x_e)] \\ [B''(x_e)] \end{Bmatrix} [Q]\{c\} \tag{42}$$

Matrix $[Q]$ is neglected temporarily from now on, when equations are derived in this section for the same reason mentioned in Section 2.4.

According to the localized property of the B-spline functions, Eq. (42) can be expressed in a simplified form:

$$\{\delta\}^e_m = [T]\{c\}^e_m \tag{43}$$

where $[T]$ is the same as Eq. (20) or Eq. (21), and

$$\{c\}^e_m = [c^e_{1,m}, c^e_{2,m}, c^e_{3,m}, c^e_{4,m}, c^e_{5,m}, c^e_{6,m}]^{\mathrm{T}} = [c_{e-3,m}, c_{e-2,m}, c_{e-1,m}, c_{e,m}, c_{e+1,m}, c_{e+2,m}]^{\mathrm{T}} \tag{44}$$

Eq. (44) shows the location of the element vector $\{c\}^e_m$ in the global vector $\{c\}_m$. Substitution of Eq. (43) into Eq. (38) gives:

$$w^e = \sum_{m=1}^{r} [H][T] Y_m(y)\{c\}^e_m = \sum_{m=1}^{r} [N] Y_m(y)\{c\}^e_m = [\overline{N}]\{c\}^e \tag{45}$$

where

$$\{c\}^e = \left[\{c\}^{e\mathrm{T}}_1, \{c\}^{e\mathrm{T}}_2, \ldots, \{c\}^{e\mathrm{T}}_r\right]^{\mathrm{T}} \tag{46}$$

$$\begin{aligned} [\overline{N}] &= \left[[\overline{N}]_1, [\overline{N}]_2, \ldots, [\overline{N}]_r\right] \\ [\overline{N}]_m &= [N] Y_m(y) \end{aligned} \tag{47}$$

in which $[N]$ is given in Eq. (23), $\{c\}^e_m$ is defined by Eq. (44). The finite strip DOF $\{c\}^e$ have no specified meanings, or say, they are generalized DOF.

3.4. Stiffness equations of the finite strip with generalized degrees of freedom

A functional of potential energy for the finite strip with generalized DOF can be derived from Eq. (45)

$$\prod^e = \frac{1}{2}\{c\}^{e\mathrm{T}}[G]^e\{c\}^e - \{c\}^{e\mathrm{T}}\{P\}^e \tag{48}$$

where

$$[G]^e = \begin{bmatrix} [G]^e_{11} & [G]^e_{12} & \cdots & [G]^e_{1r} \\ [G]^e_{21} & [G]^e_{22} & \cdots & [G]^e_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ [G]^e_{r1} & [G]^e_{r2} & \cdots & [G]^e_{rr} \end{bmatrix}; \qquad \{P\}^e = \begin{Bmatrix} \{P\}^e_1 \\ \{P\}^e_2 \\ \vdots \\ \{P\}^e_r \end{Bmatrix} \tag{49}$$

$$\begin{aligned} [G]^e_{ij} &= \textstyle\int_0^{l_y} \int_0^1 [E]^{\mathrm{T}}_i [D]^e [E]_j h \,\mathrm{d}\xi \,\mathrm{d}y \\ \{P\}^e_i &= \textstyle\int_0^{l_y} \int_0^1 [\overline{N}]^{\mathrm{T}}_i q^e h \,\mathrm{d}\xi \,\mathrm{d}y \end{aligned} \qquad \text{for } i,j = 1,2,\ldots,r \tag{50}$$

in which

$$[E]_i = \begin{bmatrix} -[B'']Y_i \\ -[B]Y_i'' \\ -2[B']Y_i' \end{bmatrix}, \qquad [D] = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \tag{51}$$

where D_x, D_y, D_{xy}, and D_1 represent the stiffness property of the plate.

The explicit form of $[G]_{ij}^e$ is:

$$[G]_{ij}^e = D_x[B^{22}]S^{00} + D_1\left([B^{02}]S^{20} + [B^{20}]S^{02}\right) + D_y[B^{00}]S^{22} + 4D_{xy}[B^{11}]S^{11} \tag{52}$$

in which $[B^{22}]$ was given explicitly in Eq. (30), and:

$$[B^{00}] = \frac{h}{3\,991\,680}\begin{bmatrix} 252 & 9113 & 29\,558 & 15\,498 & 1018 & 1 \\ & 397\,416 & 1\,558\,706 & 1\,072\,186 & 121\,641 & 1018 \\ & & 7\,464\,456 & 6\,602\,476 & 1\,072\,186 & 15\,498 \\ & & & 7\,464\,456 & 1\,558\,706 & 29\,558 \\ & & \text{symmetry} & & 397\,416 & 9113 \\ & & & & & 252 \end{bmatrix}$$

$$[B^{02}] = [B^{20}]^{\mathrm{T}} = \frac{1}{362\,880h}\begin{bmatrix} 56 & 209 & -460 & 70 & 124 & 1 \\ 2225 & 11\,996 & -20\,986 & -2540 & 8929 & 376 \\ 7856 & 60\,158 & -80\,008 & -47\,276 & 54\,664 & 4606 \\ 4606 & 54\,664 & -47\,276 & -80\,008 & 60\,158 & 7856 \\ 376 & 8929 & -2540 & -20\,986 & 11\,996 & 2225 \\ 1 & 124 & 70 & -460 & 209 & 56 \end{bmatrix} \tag{53}$$

$$[B^{11}] = \frac{1}{362\,880h}\begin{bmatrix} 70 & 1051 & 460 & -1330 & -250 & -1 \\ & 20\,638 & 19\,726 & -30\,220 & -10\,945 & -250 \\ & & 47\,248 & -35\,884 & -30\,220 & -1330 \\ & & & 47\,248 & 19\,726 & 460 \\ & & \text{symmetry} & & 20\,638 & 1051 \\ & & & & & 70 \end{bmatrix}$$

$$S^{pq} = \int_0^{l_y} Y_i^p(y)Y_j^q(y)\,\mathrm{d}y, \quad \text{for } p,q = 0,1,2 \tag{54}$$

where $Y_i^p(y)$, for $p = 0,1,2$, denotes the pth derivative of Y_i with respect to y.

The overall potential energy of the plate, $\prod$, is equal to the sum of the potential energy $\prod^e$ of all elements:

$$\prod = \sum_e \prod^e = \frac{1}{2}\{c\}^{\mathrm{T}}[G]\{c\} - \{c\}^{\mathrm{T}}\{P\} \tag{55}$$

where

$$[G] = \begin{bmatrix} [G]_{11} & [G]_{12} & \cdots & [G]_{1r} \\ [G]_{21} & [G]_{22} & \cdots & [G]_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ [G]_{r1} & [G]_{r2} & \cdots & [G]_{rr} \end{bmatrix}, \qquad \{P\} = \begin{Bmatrix} \{P\}_1 \\ \{P\}_2 \\ \vdots \\ \{P\}_r \end{Bmatrix} \tag{56}$$

Now, the transformation matrix $[Q]$ that was neglected before for the sake of simplicity, should be incorporated into Eq. (55). That is to say, $\{c\}$ in Eq. (55) should be replaced by $([Q]\{c\})$:

$$\begin{aligned}\prod = \sum_e \prod^e &= \frac{1}{2}\{c\}^{\mathrm{T}}[Q]^{\mathrm{T}}[G][Q]\{c\} - \{c\}^{\mathrm{T}}[Q]^{\mathrm{T}}\{P\} \\ &= \frac{1}{2}\{c\}^{\mathrm{T}}[K]\{c\} - \{c\}^{\mathrm{T}}\{F\}\end{aligned} \tag{57}$$

in which:

$$[K] = [Q]^{\mathrm{T}}[G][Q], \qquad \{F\} = [Q]^{\mathrm{T}}\{P\} \tag{58}$$

where $[G]$ and $\{P\}$ are overall characteristic matrices and overall characteristic vector, respectively, which are generated by means of assembling all the characteristic matrices and the characteristic vectors of the strip, respectively. Locations of the submatrix $[G]_{ij}^e$ in the submatrix $[G]_{ij}$ can be identified in terms of the relationship between $\{c\}_m^e$ and $\{c\}_m$, as shown in Eq. (44).

Based on the variational principle, the characteristic equations of the plate can be derived from Eq. (57):

$$[K]\{c\} = \{F\} \tag{59}$$

It can be concluded that the total number of discretization DOF depends on the number of DOF of the global displacement field rather than the local field of the finite element/strip when the method of finite element/strip with generalized DOF is applied to structural analysis. Moreover, there is only one DOF per inner nodal line corresponding to every term of the series in the global displacement field. Therefore, the method of finite element/strip with generalized DOF will result in the great reduction of the number of unknown coefficients.

It is well known that the quintic element/strip can attain higher accuracy than the cubic element/strip with the same division in the bending computation of the plates or beams. On the other hand, the number of DOF at every inner nodal line for the quintic B-spline displacement field is the same as that for the cubic B-spline displacement field. Therefore, the quintic finite element/strip with generalized DOF will provide us a more effective tool for structural analysis.

4. Numerical examples

The quadratic finite element or finite strip with generalized DOF can be used for analysis of various thin-walled structures. The accuracy, efficiency and versatility of the present method will be demonstrated through the following examples.

4.1. Application of the finite element with generalized degrees of freedom

The first example considers a simply-supported elastic straight beam of uniform cross section, as shown in Fig. 1, with length l, modulus of elasticity E. The beam is subjected to a uniformly distributed load, q. The quintic finite element with generalized DOF is employed to calculate the stress resultants and deflection of the beam. Results are compared with those obtained by the cubic spline Ritz method (Yang, 1998a) and the analytical method, as shown in Table 1. Here, n denotes the number of discretized elements, the abbreviation of SRM represents the spline Ritz method. I denotes the second moment of area. Solutions of these three methods show that the quintic finite element with generalized DOF attains more satisfying results with fewer discretization DOF than the cubic spline methods. It should be noted that high accuracy, as Table 1 shows, is still attained, even though a technique of subtraction of statically equivalent forces

Table 1
Deflection (w) and bending moments (M) at the central point of the beam with uniform cross-section

	n	DOF	w	M
Cubic	8	11	0.01302	0.1263
SRM	16	19	0.01302	0.1253
Present	6	11	0.01302	0.1250
Exact			0.01302	0.1250
Multiplier			ql^4/EI	ql^2

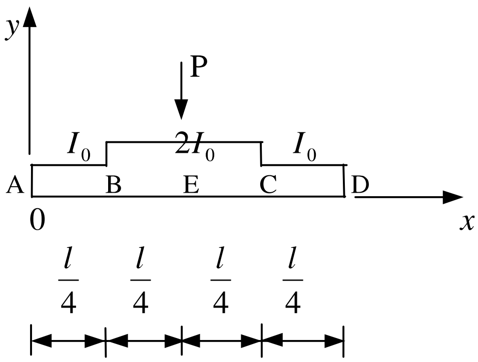

Fig. 3. The beam with discontinuity in thickness.

from solutions of stress resultants, which is adopted by the classical beam elements to improve the accuracy of the solution, is not employed in the SRM and the present element.

A stepped elastic straight beam with length l, Young's modulus E is considers as the second example. A concentrated load, P, is applied to the center point of the beam, as shown in Fig. 3. The second moment of area of the beam's cross-section about its neutral axis, I, which changes abruptly along the x direction, is similar with the cross-sectional depth. For segment AB and CD, $I = I_0$; and $I = 2I_0$ for segment CD. The quintic finite element with generalized DOF is employed to calculate the stress resultants and deflection of the beam. And the results are compared with those obtained by the conventional FEM and the exact solutions, which exhibits satisfactory agreement, as shown in Table 2. n denotes the number of the discretized element.

The third example considers an elastic straight beam with length l, Young's modulus E. A concentrated load, P, and uniformly distributed load, q, is applied to the beam, respectively. The cross-sectional depth of the beam, $h(x)$, varies linearly along the beam's axis, as shown in Fig. 4, and $h(x = 0) = h_0$, $h(x = l) = 2h_0$. According to the methodology of the conventional finite element, the cross-sectional depth of every element

Table 2
Deflections (w) and bending moments (M) of the beam with non-uniform cross section

Method	n	DOF	Points B and C		Point E	
			w	M	w	M
Present	8	13	0.00834	0.1254	0.0117	0.248
FEM	8	18	0.00833	0.1254	0.0116	0.248
Exact		0.00846	0.125	0.0117	0.250	
Multiplier			Pl^3/EI_0	Pl	Pl^3/EI_0	Pl

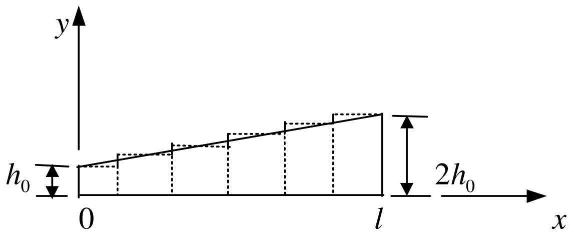

Fig. 4. Division and approximation of the beam.

Table 3
Deflection and bending moments at the central point of the beam with variable depth of cross-section

Method	Uniformly distributed load, q		Concentrated load, P	
	w	M	w	M
Present	0.004195	0.125	0.0066	0.25
FEM	0.004191	0.125	0.0066	0.244
Exact	0.004196	0.125	0.00662	0.25
Multiplier	ql^4/EI_0	ql^2	Pl^3/EI_0	Pl

is constant. So a stepped thickness of the beam is used to approximate the linearly varying one. The quintic finite element with generalized DOF is employed to calculate the stress resultants and deflection of the beam. Results are compared with those obtained by the conventional FEM and the exact solutions, which exhibits the desirable agreement, as shown in Table 3. I_0 denotes the value of I at the leftmost end of the beam in this table.

The conventional quintic finite elements and the spline finite elements are effective in the analysis of regular beams, but awkward when applied to beams with non-uniform cross-sections. From the three examples as aforementioned, it is concluded that the quintic finite element with generalized coefficients could be effectively applied not only to regular beams, but also to non-uniform beams with high accuracy,

4.2. Application of finite strip with generalized degrees of freedom

The fourth example considers a square thin plate of dimension $l \times l$, as shown in Fig. 2, with constant thickness, carrying uniformly distributed loading q. And all the four edges of the plate are simply supported. The finite strip presented in this paper and the conventional FSM (Cheung, 1976) are employed to compute the stress resultants and deflection of the plate. Results are compared with those obtained by the exact solutions, which shows a desirable agreement, as demonstrated in Table 4. m in Table 4 denotes the series term.

Table 4
Deflection and bending moments at the central point of the thin slab with uniform cross section

	The present method			Conventional FSP	Exact	Multiplier
	$m=1$	$m=3$	$\sum_{m=1}^{3}$			
w	0.00411	−0.00005	0.00406	0.004059	0.004062	q_0l^4/EI
M_x	0.04937	−0.0015	0.04787	0.00478	0.047886	q_0l^2
M_y	0.05178	−0.0045	0.04728	0.00478	0.047886	q_0l^2

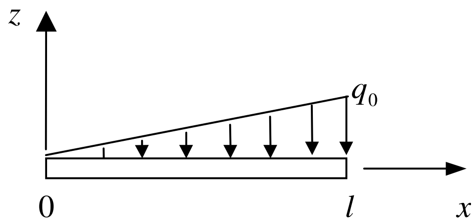

Fig. 5. A plate under hydrostatic load.

Table 5
Deflection and bending moments at the central point of the thin slab underhydrostatic load

	The present method			Exact	Multiplier
	$m=1$	$m=3$	$\sum_{m=1}^{3}$		
w	0.001525	−0.000025	0.0015	0.0015	$q_0 l^4/EI$
M_x	0.02078	−0.00078	0.02022	0.02	$q_0 l^2$
M_y	0.019938	−0.00201	0.01797	0.018	$q_0 l^2$

The fifth example considers a square thin plate with dimension $l \times l$, constant thickness, and carrying hydrostatic load, as shown in Fig. 5. One edge of the plate, where $x = 0$, is clamped, and the other three edges are simply supported. The method of finite strip with generalized DOF is employed to compute the stress resultants and deflection of the plate. And the results are compared with the exact solutions, which shows the desirable agreement, as illustrated in Table 5. m denotes the series term.

The sixth example considers a flat rectangular plate of dimension $l \times l$ with Young's modulus E, and carrying linearly distributed loading $q(x)$. All the four edges of the plate are simply supported. Its thickness varies, along the x direction, according to an exponential expression so that the distribution of the flexural rigidity of the plate EI_0 varies linearly in the same direction, as shown in Fig. 6. The finite strip with generalized DOF is employed to calculate the stress resultants and deflections of the plate, and only the first term of the series, $m = 1$, is adopted. The results are compared with those obtained by the conventional FSM (Melosh, 1961) and the exact solutions, which shows satisfactory agreement, as illustrated in Fig. 7 and Table 6.

Examples 4–6 demonstrated that the quintic finite strip with generalized coefficients can be effectively applied to both uniform and non-uniform plates with high accuracy.

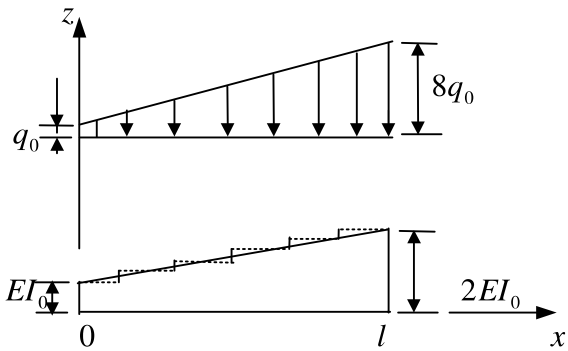

Fig. 6. Distribution of flexural rigidity of the plate under linearly distributed load.

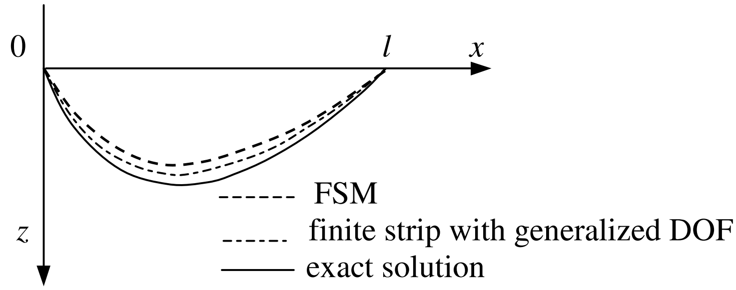

Fig. 7. Deflection of the plate at its central line alone x direction.

Table 6
Deflection and bending moments of the thin plate with variable thickness at its central point

	Present method	FSP	FEM	Exact	Multiplier
w	0.004273	0.004276	0.004227	0.004266	ql^4/EI_0
M_x	0.1897	0.189	0.186	0.1913	ql^2
M_y	0.2154	0.0215	0.1969	0.2158	ql^2

5. Conclusions

This paper presents an efficient and versatile quintic finite element/strip with generalized DOF. The total number of discretised DOF depends on the number of DOF of the global displacement field rather than the local field of the element/strip when the method of finite element/strip with generalized DOF is applied to structural analysis. Moreover, there is only one DOF per inner nodal line corresponding to every term of the series in the global displacement field. Therefore, the method of finite element/strip with generalized DOF will result in a great reduction of the number of unknown coefficients.

The quintic finite element/strip with generalized DOF can provides us a more effective tool for structural analysis, as compared with the cubic spline function methods and the cubic finite elements. And the quintic element/strip can attain higher accuracy than the cubic element/strip with the same discretization mesh in bending computation of a plate or a beam.

The quintic finite element/strip with generalized DOF can yield without doubt desirable results of greater accuracy with much fewer number of DOF, like the conventional quintic finite element/strip, in the cases where the steep gradient of field variable is expected. Even if the gradient of field variable is steady, the present method of quintic finite element/strip with generalized DOF will reduce the number of discretization DOF and hence the computational time, unlike the classical quintic element or strip, which will not reduce DOF and the computational time.

This kind of element or strip presented in this paper not only inherits some of the advantages of the conventional spline function methods and the conventional quintic finite element/strip with higher accuracy and fewer DOF, but also overcomes their disadvantage when analyzing beams, plates and other structures with non-uniform distribution of thickness.

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